Chapter 1: What is maths? And why do we *all* need it?

From *The Elephant in the Classroom: Helping Children Learn & Love Maths* by Jo Boaler, published by Souvenir Press, 2008.

In my different research studies I have asked hundreds of children, taught traditionally, to tell me what maths is. They will typically say such things as "numbers" or "lots of rules". Ask mathematicians what maths is and they will more typically tell you that it is "the study of patterns" or that it is a "set of connected ideas". Students of other subjects, such as English and science, give similar descriptions of their subjects to experts in the same fields. Why is maths so different? And why is it that students of maths develop such a distorted view of the subject?

Reuben Hersh, a philosopher and mathematician, has written a book called 'What is Mathematics, Really?' in which he explores the true nature of mathematics and makes an important point - people don't like mathematics because of the way it is *mis-represented* in school. The maths that millions of school children experience is an impoverished version of the subject that bears little resemblance to the mathematics of life or work, or even the mathematics in which mathematicians engage.

What is mathematics, really?

Mathematics is a human activity, a social phenomenon, a set of methods used to help illuminate the world, and it is part of our culture. In Dan Brown's best-selling novel *The DaVinci Code*[i], the author introduces readers to the 'divine proportion,' a ratio that is also known as the Greek letter phi. This ratio was first discovered in 1202 when Leonardo Pisano, better known as Fibonacci, asked a question about the mating behavior of rabbits. He posed this problem:

> *A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?*

The resulting sequence of pairs of rabbits, now known as the Fibonacci sequence, is

$$1, 1, 2, 3, 5, 8, 13, \ldots$$

Moving along the sequence of numbers, dividing each number by the one before it, produces a ratio that gets closer and closer to 1.618, also known as phi, or *the golden ratio*. What is amazing about this ratio is that it exists throughout nature. When flower seeds grow in spirals they grow in the ratio 1.618:1. The ratio of spirals in seashells, pinecones and pineapples is exactly the same. For example, of you look very carefully at the photograph of a daisy below you will see that the seeds in the center of the flower form spirals, some of which curve to the left and some to the right.
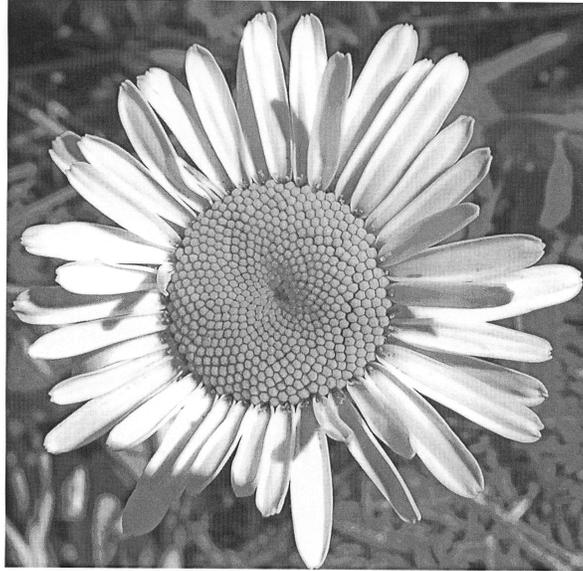
If you map the spirals carefully you will see that close to the center there are 21 running anti-clockwise. Just a little further out there are 34 spirals running clockwise. These numbers appear next to each other in the Fibonnacci sequence.

| Daisy showing 21 anti-clockwise spirals. | Daisy showing 34 clockwise spirals |
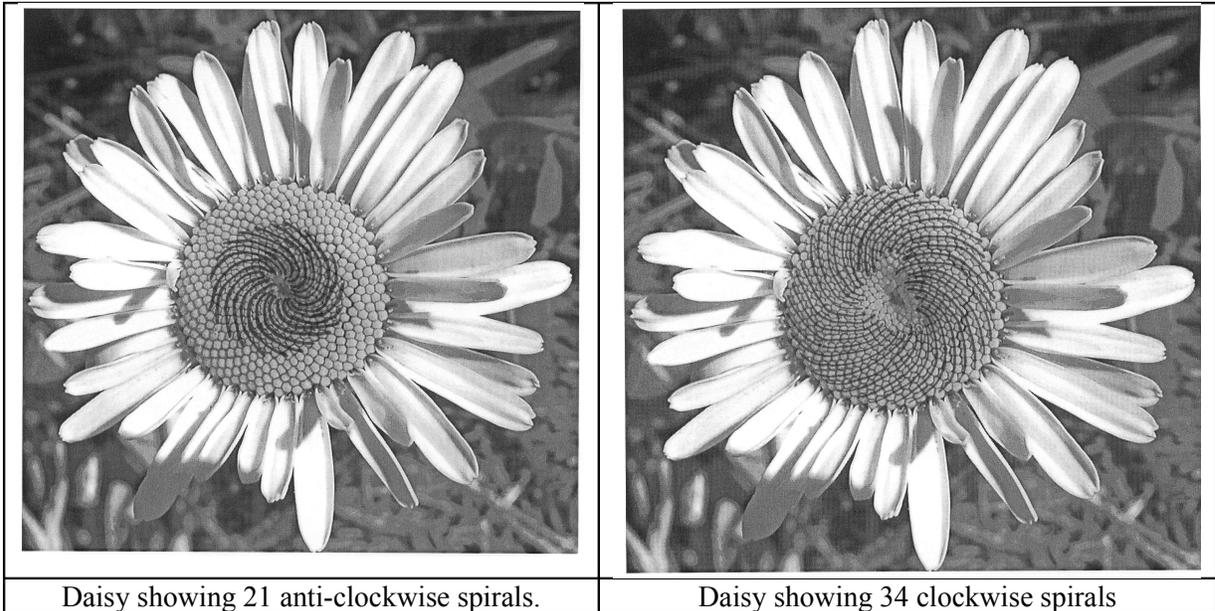
Remarkably, the measurements of various parts of the human body have the exact same relationship. Examples include a person's height divided by the distance from tummy button to

the floor; or the distance from shoulders to finger-tips, divided by the distance from elbows to finger-tips. The ratio turns out to be so pleasing to the eye that it is also ubiquitous in art and architecture, featuring in the United Nations Building, the Greek Parthenon, and the pyramids of Egypt.

Ask most mathematics students in secondary schools about these relationships and they will not even know they exist. This is not their fault of course, they have never been taught about them. Mathematics is all about illuminating relationships such as those found in shapes and in nature. It is also a powerful way of expressing relationships and ideas in numerical, graphical, symbolic, verbal and pictorial forms. This is the wonder of mathematics that is denied to most children.

Those children who do learn about the true nature of mathematics are very fortunate and it often shapes their lives. Margaret Wertheim, a science reporter for *The New York Times*, reflects upon an Australian mathematics classroom from her childhood and the way that it changed her view of the world:

> When I was ten years old I had what I can only describe as a mystical experience. It came during a math class. We were learning about circles, and to his eternal credit our teacher, Mr Marshall, let us discover for ourselves the secret image of this unique shape: the number known as pi. Almost everything you want to say about circles can be said in terms of pi, and it seemed to me in my childhood innocence that a great treasure of the universe had just been revealed. Everywhere I looked I saw circles, and at the heart of every one of them was this mysterious number. It was in the shape of the sun and the moon and the earth; in mushrooms, sunflowers, oranges, and pearls; in wheels, clock faces, crockery, and telephone dials. All of these things were united by pi, yet it transcended them all. I was enchanted. It was as if someone had lifted a veil and shown me a glimpse of a marvelous realm beyond the one I experienced with my senses. From that day on I knew I wanted to know more about the mathematical secrets hidden in the world around me.[ii]

How many students who have sat through maths classes would describe mathematics in this way? Why are they not enchanted, as Wertheim was, by the wonder of mathematics, the insights it provides into the world, the way it elucidates the patterns and relationships all around us? It is because they are misled by the image of maths presented in school mathematics classrooms and they are not given an opportunity to experience real mathematics. Ask most school students what

maths is and they will tell you it is a list of rules and procedures that need to be remembered.[iii] Their descriptions are frequently focused on calculations. Yet as Keith Devlin, mathematician and writer of several books about maths points out, mathematicians are often not even very good at calculations as they do not feature centrally in their work. Ask mathematicians what maths is and they are more likely to describe it as *the study of patterns*.[iv] [v]

Early in his book 'The Math Gene' Devlin tells us that he hated maths in his English primary school. He then recalls his reading of W.W. Sawyer's book 'Prelude to Mathematics' during secondary school that captivated his thinking and even made him start considering becoming a mathematician himself. Devlin quotes the following from Sawyer's book:

> ' "Mathematics is the classification and study of all possible patterns." Pattern is here used in a way that everybody may agree with. It is to be understood in a very wide sense, to cover almost *any kind of regularity that can be recognized by the mind.* Life, and certainly intellectual life, is only possible because there are certain regularities in the world. A bird recognizes the black and yellow bands of a wasp; man recognizes that the growth of a plant follows the sowing of a seed. In each case, a mind is aware of pattern.'[vi]

Reading Sawyer's book was a fortunate event for Devlin, but insights into the true nature of mathematics should not be gained *in spite* of school experiences, nor should they be left to the few who stumble upon the writings of mathematicians. I will argue, as others have done before me, that school classrooms should give children a sense of the nature of mathematics, and that such an endeavor is critical in halting the low achievement and participation that is so commonplace. School children know what English literature and science are because they engage in authentic versions of the subjects in school. Why should mathematics be so different?[vii]

What do mathematicians do, really?

Fermat's Last Theorem, as it came to be known, was a theory proposed by the great French mathematician, Pierre de Fermat, in the 1630's. Proving (or disproving) the theory that Fermat set out became the challenge for centuries of mathematicians and caused the theory to become known as "the world's greatest mathematical problem."[viii] Fermat was born in 1603 and was famous in his time for posing intriguing puzzles and discovering interesting relationships between numbers. Fermat claimed that the equation $a^n + b^n = c^n$ has no solutions for n when n is greater

than 2 and a non zero integer. So, for example, no numbers could make the statement $a^3 + b^3 = c^3$ true. Fermat developed his theory through consideration of Pythagoras' famous case of $a^2 + b^2 = c^2$. School children are typically introduced to the Pythagorean formula when learning about triangles, as any right-angled triangle has the property that the sum of squares built on the two sides ($a^2 + b^2$) is equal to the square of the hypotenuse c^2. So, for example, when the sides of a triangle are 3 and 4 then the hypotenuse must be 5 because $3^2 + 4^2 = 5^2$. Sets of three numbers that satisfy Pythagoras' case are those where two square numbers (eg 4, 9, 16, 25) can combine to produce a third.

Fermat was intrigued by the Pythagorean triples and explored the case of cube numbers, reasonably expecting that some pairs of cubed numbers could be combined to produce a third cube. But Fermat found this was not the case and the resulting cube always has too few or too many blocks, for example:

| 9^3 | + | 10^3 | $\neq$ | 12^3 |
| 729 | + | 1000 | $\neq$ | 1728 |
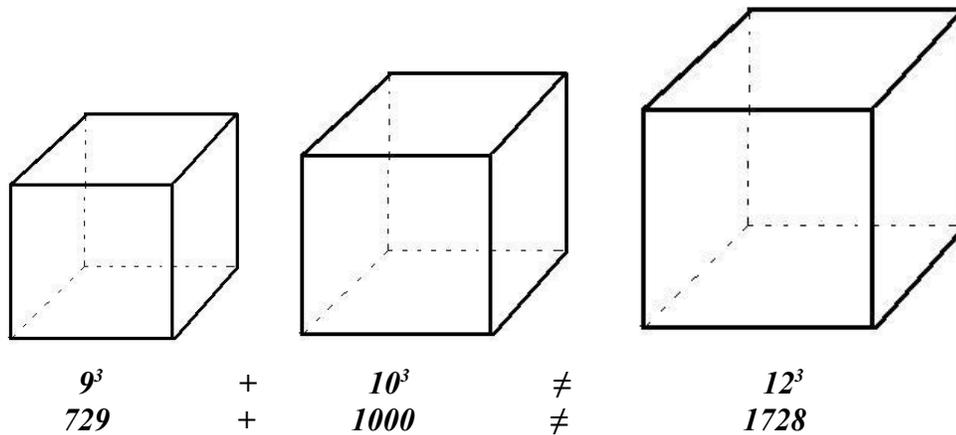
The sum of the volumes of cubes of dimension 9 and 10 almost equals the volume of a cube of dimension 12, but not quite (it is one short!).

Indeed Fermat went on to claim that even if every number in the world was tried, no-one would ever find a solution to $a^3 + b^3 = c^3$ nor to $a^4 + b^4 = c^4$, or any higher power. This was a bold claim involving the universe of numbers. In mathematics it is not enough to make such claims, even if the claims are backed up by hundreds of cases, as mathematics is all about the construction of time-resistant proofs. Mathematical proofs involve making a series of logical statements from which only one conclusion can follow and, once constructed, they are always true. Fermat made an important claim in 1630 but he did not provide a proof and it was the proof of his claim that would elude and frustrate mathematicians for over 350 years. Not only did Fermat not provide a

proof but he scribbled a note in the margin of his work saying that he had a "marvelous" proof of his claim but that there was not enough room to write it. This note tormented mathematicians for centuries as they tried to solve what some have claimed to be *the world's greatest mathematical problem.*[ix]

'Fermat's last theorem' stayed unsolved for over 350 years, despite the attentions of some of the greatest minds in history. In recent years it was dramatically solved by a shy English mathematician, and the story of his work, told by a number of biographers, captures the drama, the intrigue and the allure of mathematics that is unknown by many. Any child – or adult – wanting to be inspired by the values of determination and persistence, enthralled by the intrigue of puzzles and questions, and introduced to the sheer beauty of living mathematics should read Simon Singh's book *Fermat's Enigma*. Singh describes 'one of the greatest stories in human thinking'[x] providing important insights into the ways mathematicians work.

Many people had decided that there was no proof to be found of Fermat's theorem and that this great mathematical problem was unsolvable. Prizes were offered from different corners of the globe and men and women devoted their lives to finding a proof, to no avail. Andrew Wiles, the mathematician who would write his name into history books, first encountered Fermat's theory as a 10 year old boy while reading in his local library in his home town of Cambridge. Wiles described how he felt when he read the problem, saying that 'It looked so simple, and yet all the great mathematicians in history could not solve it. Here was a problem that I, as a ten-year-old, could understand and I knew from that moment that I would never let it go, I had to solve it.'[xi] Years later Wiles graduated with a PhD in mathematics from Cambridge and then moved to Princeton to take a position in the mathematics department. But it was still some years later when Wiles realized that he could devote his life to the problem that had intrigued him since childhood.

As Wiles set about trying to prove Fermat's Last Theorem he retired to his study and started reading journals, gathering new techniques. He started exploring and looking for patterns, working on small areas of mathematics and then standing back to see if they could be illuminated by broader concepts. Wiles worked on a number of different techniques over the next few years, exploring different methods for attacking the problem. Some seven years after starting the problem Wiles emerged from his study one afternoon and announced to his wife that he had solved Fermat's Last Theorem.

The venue that Wiles chose to present his proof of the 350 year-old problem was a conference at the Isaac Newton Institute in Cambridge, England in 1993. Some people had become intrigued about Wiles' work and rumors had started to filter through that he was actually going to present a proof of Fermat's Last Theorem. By the time Wiles came to present his work there were over two hundred mathematicians crammed into the room, and some had sneaked in cameras to record the historic event. Others – who could not get in – peered through windows. Wiles needed three lectures to present his work and at the conclusion of the last lecture the room erupted into great applause. Singh described the atmosphere of the rest of the conference as 'euphoric' with the world's media flocking to the Institute. Was it possible that this great and historical problem had finally been solved? Barry Mazur, a number theorist and algebraic geometer, reflected on the event saying that 'I've never seen such a glorious lecture, full of such wonderful ideas, with such dramatic tension, and what a build up. There was only one possible punch line.' Everyone who had witnessed the event thought that Fermat's Last Theorem was finally proved. Unfortunately, there was an error in Wiles' proof that meant that Wiles had to plunge himself back into the problem. In September 1994, after a few more months of work, Wiles knew that his proof was complete and correct. Using many different theories, making connections that had not previously been made, Wiles had constructed beautiful new mathematical methods and relationships. Ken Ribet, a Berkeley mathematician whose work had contributed to the proof, concluded that the landscape of mathematics had changed and mathematicians in related fields could work in ways that had never been possible before.

The story of Wiles is fascinating and told in more detail by Simon Singh and others. But what do such accounts tell us that could be useful in improving children's education? One clear difference between the work of mathematicians and schoolchildren is that mathematicians work on long and complicated problems that involve combining many different areas of mathematics. This stands in stark contrast to the short questions that fill the hours of maths classes and that involve the repetition of isolated procedures. Long and complicated problems are important to work on for many reasons, one of them being that they encourage persistence, one of the values that is critical for young people to develop and that will stand them in good stead in life and work. When mathematicians are interviewed they often speak of the enjoyment they experience from working on difficult problems. Diane Maclagan, a professor at Rutgers University in the US, was asked: what is the most difficult aspect of your life as a mathematician? She replied "Trying to prove theorems". And the most fun? the interviewer asked. "Trying to prove theorems." She replied.[xii] Working on long and complicated problems may not sound like fun, but mathematicians find

such work enjoyable because they are often successful. It is hard for any school child to enjoy a subject if they experience repeated failure, which of course is the reality for many young people in school mathematics classrooms. But the reason that mathematicians are successful is because they have learned something very important – and very learnable. They have learned to problem solve.

Problem solving is at the core of mathematician's work, as well as the work of engineers and others, and it starts with the making of a guess. Imre Lakatos, mathematician and philosopher, describes mathematical work as 'a process of "conscious guessing" about relationships among quantities and shapes'[xiii]. Those who have sat in traditional maths classrooms are probably surprised to read that mathematicians highlight the role of guessing, as I doubt whether they have *ever* experienced any encouragement to guess in their maths classes. When an official report in the UK was commissioned to examine the mathematics needed in the workplace the reviewers found that estimation was the most useful mathematical activity.[xiv] Yet when children who have experienced traditional maths classes are asked to estimate they are often completely flummoxed and try to work out exact answers then round them off to look like an estimate. This is because they have not developed a good *feel* for numbers, which would allow them to estimate instead of calculate, and also because they have learned, wrongly, that mathematics is all about precision, not about making estimates or guesses. Yet both are at the heart of mathematical problem solving.

After making a guess mathematicians engage in a zig-zagging process of conjecturing, refining with counter-examples, and then proving. Such work is exploratory and creative and many writers draw parallels between mathematical work and art or music. Robin Wilson, a British mathematician, proposes that mathematics and music 'are both creative acts. When you are sitting with a bit of paper creating mathematics, it is very like sitting with a sheet of music paper creating music.'[xv] Devlin agrees saying that 'Mathematics is not about numbers, but about life. It is about the world in which we live. It is about ideas. And far from being dull and sterile, as it is so often portrayed, it is full of creativity.'[xvi]

The exhilarating, creative pathways that mathematicians describe as they solve problems, often hidden in the end-point of mathematical work, cannot be the exact same pathways that school children experience, as children need to be taught the methods they need, as well as use them in the solving of problems, but neither should school mathematics be so different as to be unrecognizable. As George Pólya, the eminent Hungarian mathematician, reflected, in 1945:

'A teacher of mathematics has a great opportunity. If he fills his allotted time with drilling his students in routine operations he kills their interest, hampers their intellectual development, and misuses his opportunity. But if he challenges the curiosity of his students by setting them problems proportionate to their knowledge, and helps them to solve their problems with stimulating questions, he may give them a taste for, and some means of, independent thinking.'[xvii] (Polya, 1971, v)

Another interesting feature of the work of mathematicians is its collaborative nature. Many people think of mathematicians as people who work in isolation, but this is far from the truth. Leone Burton, a professor of mathematics education, interviewed 70 research mathematicians and found that they generally challenged the solitary stereotype of mathematical work, reporting that they preferred to collaborate in the production of ideas. Over half of the papers they submitted to Burton as representative of their work were written with colleagues. The mathematicians interviewed gave many reasons for collaboration, including the advantage of learning from one another's work, increasing the quality of ideas, and sharing the 'euphoria' of problem solving. As Burton reflected, 'they offered all the same reasons for collaborating on research that are to be found in the educational literature advocating collaborative work in classrooms.'[xviii] Yet non-collaborative maths classrooms continue to prevail across England.

Something else that we learn from various accounts of mathematicians' work is that an important part of real, living mathematics is the posing of problems. Viewers of *A Beautiful Mind* may remember John Nash (played by Russell Crowe) undergoing an emotional search to form a question that would be sufficiently interesting to be the focus of his work. People commonly think of mathematicians as solving problems but as Peter Hilton[xix], an algebraic topologist, has said 'Computation involves going from a question to an answer. Mathematics involves going from an answer to a question.' Such work requires creativity, original thinking, and ingenuity. All the mathematical methods and relationships that are now known and taught to school children started as questions, yet students do not see the questions. Instead they are taught content that often appears as a long list of answers to questions that nobody has ever asked. Reuben Hersh, an American mathematician, puts it well:

'The mystery of how mathematics grows is in part caused by looking at mathematics as answers without questions. That mistake is made only by people who have had no contact with mathematical life. It's the questions that drive mathematics. Solving

problems and making up new ones is the essence of mathematical life. If

mathematics is conceived apart from mathematical life, of course it seems – dead.'[xx]

Bringing mathematics back to life for school children involves giving them a sense of living mathematics. When school students are given opportunities to ask their own questions and to extend problems into new directions, they know mathematics is still alive, not something that has already been decided and just needs to be memorized. Posing and extending problems of interest to students mean they enjoy mathematics more, they feel more ownership of their work and they ultimately learn more. English school children used to work on long problems that they could extend into directions that were of interest to them in maths classes. For example, in one problem students were asked to design any type of building. This gave them the opportunity to consider interesting questions involving high-level mathematics, such as the best design for a fire station with a firefighter's pole. Teachers used to submit the students' work to examination boards and it was assessed as part of the students' final grade. When I asked English school children about their work on these problems they not only reported that they were enjoyable and they learned a lot from them, but that their work made them "feel proud" and that they could not feel proud of their more typical textbook work. Mathematical coursework no longer exists in England as it was decided that it too often led to cheating. Unfortunately this was one of few experiences schoolchildren had to use maths in the solving of real and interesting problems.

Another important part of the work of mathematicians that enables successful problem solving is the use of a range of representations such as symbols, words, pictures, tables and diagrams, all used with extreme precision. The precision required in mathematics has become something of a hallmark for the subject and it is an aspect of mathematics that both attracts and repels. For some school children it is comforting to be working in an area where there are clear rules for ways of writing and communicating. But for others it is just too hard to separate the precision of mathematical language with the uninspiring "drill and kill" methods that they experience in their maths classrooms. There is no reason that precision and drilled teaching methods need to go together and the need for precision with terms and notation does not mean that mathematical work precludes open and creative exploration. On the contrary, it is the fact that mathematicians can rely on the precise use of language, symbols and diagrams that allows them to freely explore the *ideas* that such communicative tools produce. Mathematicians do not play with the notations, diagrams, and words as a poet or artist might, instead they explore the relations and insights that are revealed by different arrangements of the notations. As Keith Devlin reflects:

'Mathematical notation no more *is* mathematics than musical notation is music. A page of sheet music represents a piece of music, but the notation and the music are not the same; the music itself happens when the notes on the page are sung or performed on a musical instrument. It is in its performance that the music comes alive; it exists not on the page but in our minds. The same is true for mathematics.'[xxi]

Mathematics is a performance, a living act, a way of interpreting the world. Imagine music lessons in which students worked through hundreds of hours of sheet music, adjusting the notes on the page, receiving ticks and crosses from the teachers, but never playing the music. Students would not continue with the subject because they would never experience what music *was*. Yet this is the situation that continues in mathematics classes, seemingly unabated.

Those who use mathematics engage in mathematical *performances*, they use language in all its forms, in the subtle and precise ways that have been described, in order to do something with mathematics. Students should not just be memorizing past methods; they need to engage, do, act, perform, problem solve, for if they don't *use* mathematics as they learn it they will find it very difficult to do so in other situations, including examinations.

The erroneous thinking behind many school approaches is that students should spend years being drilled in a set of methods that they can use later. Many mathematicians are most concerned about the students who will enter post-graduate programs in mathematics. At that point students will encounter real mathematics and use the tools they have learned in school to work in new, interesting and authentic ways. But by this time most maths students have given up on the subject. We cannot keep pursuing an educational model that leaves the best and the only real taste of the subject to the end, for the rare few who make it through the grueling eleven years that precede it. If students were able to work in the ways mathematicians do, for at least some of the time – posing problems, making guesses and conjectures, exploring with and refining ideas, and discussing ideas with others, then they would not only be given a sense of true mathematical work, which is an important goal in its own right,[xxii] they would also be given the opportunities to enjoy mathematics and learn it in the most productive way.[xxiii] [xxiv] [xxv]

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[ix] Ibid.

[x] Singh, S. (1997) *Fermat's Enigma: The Epic Quest to solve the world's greatest mathematical problem*. New York: Anchor Books. P. xiii

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[xix] *Peter Hilton, popular quote.*

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